Full Vectorial Finite Element Formalism for Lossy Anisotropic Waveguides

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Abstract —An efficient computer-aided solution procedure based on the finite-element method is developed for solving general waveguiding structures containing lossy anisotropic materials. In this procedure, a formulation in terms of the transverse magnetic field component is adopted and the eigenvalue of the final matrix equation corresponds to the propagation constant itself. Thus one can avoid the unnecessary iterations which arise when using complex frequencies. To demonstrate the strength of the present method, numerical results are shown for a rectangular waveguide filled with lossy anisotropic dielectric with off-diagonal elements in a permittivity tensor.

I. Introduction

COMPUTER-AIDED numerical analysis has become a necessary tool for designing microwave and optical waveguiding structures such as image guide, microstrip line, optical channel guide, and optical fiber. Increasing complexities of modern wave functional devices, particularly in monolithic integrated circuit form, have created a critical need for more accurate and efficient computer-aided analysis techniques [1].

Of the methods available, the finite element method (FEM) enables one to predict accurately the modal characteristics of a waveguide system with an arbitrary cross section. Most of the applications of the FEM to date have been focused on a loss-free system. Recently, a formalism in terms of the transverse magnetic field component established for a loss-free system [2]–[4] has been extended by the authors to a lossy system [5], [6]. The main advantage of this approach is that one can avoid the unnecessary iterations which arise when using complex frequencies [7] because the eigenvalue of the final matrix equation to be solved corresponds to the propagation constant itself. However, it was assumed in [5] and [6] that the permittivity tensor has no off-diagonal elements.

In this paper, the approach of [5] and [6] is extended to waveguides containing arbitrary lossy anisotropic materials with off-diagonal elements in a permittivity tensor. To demonstrate the strength of this method, numerical results for a rectangular waveguide filled with lossy anisotropic dielectric are presented and compared with those obtained by the telegrapher equation method [8].

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II. Finite Element Formalism

We consider a three-dimensional dielectric waveguide with an arbitrary cross section Ω in the xy plane (Fig. 1) whose relative permittivity tensor $[\epsilon]$ is arbitrary:

$$[\epsilon]=\begin{bmatrix}\epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz}\\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz}\\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz}\end{bmatrix} \tag{1}$$

$$\epsilon_{ij}=\epsilon'_{ij}-j\epsilon''_{ij} \tag{2}$$

where ϵ'_{ij} and ϵ''_{ij} are the real and the imaginary part of the complex relative permittivity ϵ_{ij}, respectively. If all the off-diagonal elements are neglected, i.e., $\epsilon_{ij}=0$ for all $i\neq j$, (1) is reduced to the diagonal tensor assumed in [5] and [6].

According to the same procedure as in the diagonal permittivity tensor [5], [6], the following matrix equation with the complex transverse magnetic field component $\{H_t\}$ is derived:

$$\left([\tilde{S}_{tt}]+k_0[\tilde{T}'_{tt}]-k_0^2[\tilde{T}_{tt}]\right)\{H_t\}=\{0\} \tag{3}$$

where

$$[\tilde{S}_{tt}]=[D]^T[S][D] \tag{4}$$

$$[\tilde{T}'_{tt}]=[D]^T[T'][D] \tag{5}$$

$$[\tilde{T}_{tt}]=[D]^T[T][D] \tag{6}$$

$$\{H_t\}=\begin{bmatrix}\{H_x\}\\ \{H_y\}\end{bmatrix}. \tag{7}$$

Here, k_0 is the free-space wavenumber, $\{0\}$ is a null vector, the superscript T denotes a transposition, and the components of vectors $\{H_x\}$ and $\{H_y\}$ are the values of H_x and H_y at nodal points in Ω, respectively. $[S]$, $[T']$, $[T]$,

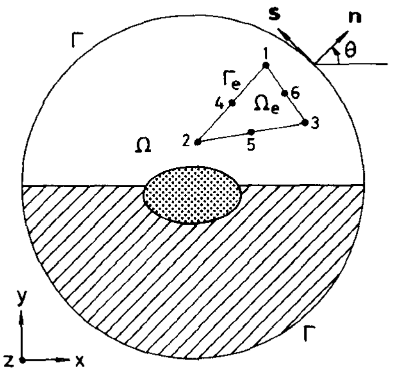

Fig. 1. Geometry of the problem. n: unit normal vector; s: unit tangential vector; θ: angle between n and x axis; Ω: waveguide cross section; Γ: impedance wall.

and $[D]$ are as follows [5], [6]:

$$[S] = \sum_e \iint_e [B'][\epsilon]_e^{-1}[B]^T \, dx\, dy \tag{8}$$

$$[T'] = j(Z_n/Z_0) \sum_{e'} \int_{e'} [N]^*[N_\theta]^T \, d\Gamma \tag{9}$$

$$[T] = \sum_e \iint_e [N]^*[N]^T \, dx\, dy \tag{10}$$

$$[B] = \begin{bmatrix} \{0\} & -\gamma\{N\} & -\{N\}_y \\ \gamma\{N\} & \{0\} & \{N\}_x \\ j\{N\}_y & -j\{N\}_x & \{0\} \end{bmatrix} \tag{11}$$

$$[B'] = \begin{bmatrix} \{0\} & \gamma\{N\} & -\{N\}_y \\ -\gamma\{N\} & \{0\} & \{N\}_x \\ -j\{N\}_y & j\{N\}_x & \{0\} \end{bmatrix} \tag{12}$$

$$[N_\theta] = \begin{bmatrix} \sin^2\theta\{N\} & -\sin\theta\cos\theta\{N\} & \{0\} \\ -\sin\theta\cos\theta\{N\} & \cos^2\theta\{N\} & \{0\} \\ \{0\} & \{0\} & j\{N\} \end{bmatrix} \tag{13}$$

$$[N] = \begin{bmatrix} \{N\} & \{0\} & \{0\} \\ \{0\} & \{N\} & \{0\} \\ \{0\} & \{0\} & j\{N\} \end{bmatrix} \tag{14}$$

$$[D] = \begin{bmatrix} [U] \\ [D_z]^{-1}[D_t] \end{bmatrix} \tag{15}$$

$$[D_z] = \sum_e \iint_e \{N\}\{N\}^T \, dx\, dy \tag{16}$$

$$[D_t] = -j\gamma^{-1} \sum_e \iint_e \left[\{N\}\{N\}_x^T \quad \{N\}\{N\}_y^T \right] dx\, dy. \tag{17}$$

Here Σ_e and $\Sigma_{e'}$ stand for summation over all elements related to the domain Ω and the boundary Γ, respectively; $\gamma = \alpha + j\beta$ is the propagation constant (α = attenuation constant, β = phase constant); Z_n and Z_0 are the surface impedance of Γ and the intrinsic impedance of vacuum, respectively; the superscript $*$ denotes a complex conjugation; $\{N\}$ is a shape function vector [4]; $\{N\}_\xi \equiv \partial\{N\}/\partial\xi$; and $[U]$ is a unit matrix. Provided that Γ is a perfect electric or magnetic wall, the second term of the left-hand side of (3) is dropped. Note that (3) is applicable to arbitrarily anisotropic waveguides whose complex relative permittivity tensor is given by (1). If all the off-diagonal elements involved in (1) are neglected, (3) is reduced to [5, eq. (20)].

Substituting (8)–(17) into (3)–(6) and ordering (3) according to a descending power of γ, we obtain

$$\left(\gamma^4[C^4] + \gamma^3[C^3] + \gamma^2[C^2] + \gamma[C^1] + [C^0]\right)\{H_t\} = \{0\}. \tag{18}$$

The explicit form of submatrices of $[C^i]$ is given in Appendix I.

Since (18) is a quadruple eigenvalue problem, it can be reduced to the following linearized form [9]:

$$([K_4] - \gamma[M_4])\{q_4\} = \{0\} \tag{19}$$

where

$$[K_4] = \begin{bmatrix} -[C^0] & [0] & [0] & [0] \\ [0] & [C^2] & [C^3] & [C^4] \\ [0] & [C^3] & [C^4] & [0] \\ [0] & [C^4] & [0] & [0] \end{bmatrix} \tag{20}$$

$$[M_4] = \begin{bmatrix} [C^1] & [C^2] & [C^3] & [C^4] \\ [C^2] & [C^3] & [C^4] & [0] \\ [C^3] & [C^4] & [0] & [0] \\ [C^4] & [0] & [0] & [0] \end{bmatrix} \tag{21}$$

$$\{q_4\} = \begin{bmatrix} \{H_t\} \\ \gamma\{H_t\} \\ \gamma^2\{H_t\} \\ \gamma^3\{H_t\} \end{bmatrix}. \tag{22}$$

Here [0] is a null matrix.

For non-self-adjoint systems, one should further reduce (19) to the standard eigenvalue problem:

$$[A_4]\{q_4\} = \gamma\{q_4\} \tag{23}$$

where

$$[A_4] = \begin{bmatrix} [0] & [U] & [0] & [0] \\ [0] & [0] & [U] & [0] \\ [0] & [0] & [0] & [U] \\ [A_{41}] & [A_{42}] & [A_{43}] & [A_{44}] \end{bmatrix} \tag{24}$$

$$[A_{41}] = -[C^4]^{-1}[C^0] \qquad [A_{42}] = -[C^4]^{-1}[C^1]$$

$$[A_{43}] = -[C^4]^{-1}[C^2] \qquad [A_{44}] = -[C^4]^{-1}[C^3].$$

For $\epsilon_{xz} = \epsilon_{zx} = \epsilon_{yz} = \epsilon_{zy} = 0$, $[C_3] = [C_1] = [0]$. In this case, (18) is simplified as

$$\left(\gamma^4[C^4] + \gamma^2[C^2] + [C^0]\right)\{H_t\} = \{0\}. \tag{25}$$

Since (25) is a quadratic eigenvalue problem, it can be reduced to the following linearized form [9]:

$$([K_2]-\gamma^2[M_2])\{q_2\}=\{0\} \tag{26}$$

where

$$[K_2]=\begin{bmatrix}-[C^0] & [0]\\ [0] & [C^4]\end{bmatrix} \tag{27}$$

$$[M_2]=\begin{bmatrix}[C^2] & [C^4]\\ [C^4] & [0]\end{bmatrix} \tag{28}$$

$$\{q_2\}=\begin{bmatrix}\{H_t\}\\ \gamma^2\{H_t\}\end{bmatrix}. \tag{29}$$

For non-self-adjoint systems, one should further reduce (26) to the standard eigenvalue problem:

$$[A_2]\{q_2\}=\gamma^2\{q_2\} \tag{30}$$

where

$$[A_2]=\begin{bmatrix}[0] & [U]\\ [A_{41}] & [A_{43}]\end{bmatrix}. \tag{31}$$

Equations (23) and (30) are standard eigenvalue problems whose eigenvalues directly correspond to the propagation constant. Thus, one can avoid unnecessary iterations using complex frequencies. The only disadvantage of them is that they involve, respectively, $8N_p$ and $4N_p$ unknown components in each eigenvector compared with $2N_p$ components in the original system, where N_p is the number of nodal points.

It is interesting to compare the results derived above with those derived from the penalty function method [10]. Considering full tensor elements (1) and ordering the matrix equation of the system according to a descending power of γ, we obtain

$$(\gamma^2[P^2]+\gamma[P^1]+[P^0])\{H\}=\{0\} \tag{32}$$

where

$$\{H\}=\begin{bmatrix}\{H_t\}\\ \{H_z\}\end{bmatrix}. \tag{33}$$

The explicit form of nonzero submatrices of $[P^i]$ is given in Appendix II.

Equation (32) can then be reduced to

$$([K]-\gamma[M])\{q\}=\{0\} \tag{34}$$

or

$$[A]\{q\}=\gamma\{q\} \tag{35}$$

where

$$[K]=\begin{bmatrix}-[P^0] & [0]\\ [0] & [P^2]\end{bmatrix} \tag{36}$$

$$[M]=\begin{bmatrix}[P^1] & [P^2]\\ [P^2] & [0]\end{bmatrix} \tag{37}$$

$$[A]=\begin{bmatrix}[0] & [U]\\ -[P^2]^{-1}[P^0] & -[P^2]^{-1}[P^1]\end{bmatrix} \tag{38}$$

$$\{q\}=\begin{bmatrix}\{H\}\\ \gamma\{H\}\end{bmatrix}. \tag{39}$$

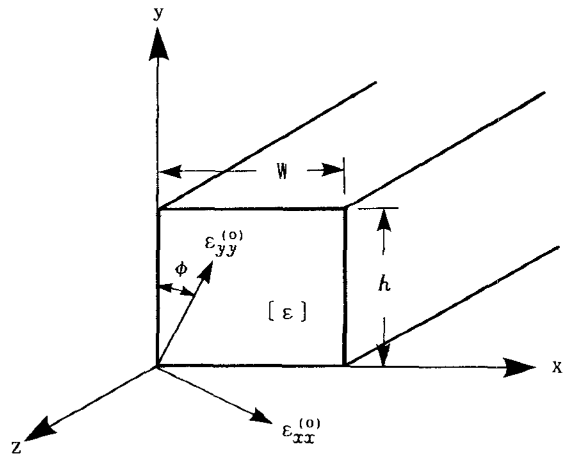

Fig. 2. Rectangular waveguide filled with lossy anisotropic material.

Note that (34) and (35) involve $6N_p$ unknown components in each eigenvector.

III. Application

In this section, we present computed results obtained by (18). In numerical implementations, the HITAC S-810/10 supercomputer is used and double precision is adopted to avoid round-off errors. Inverse matrices are computed via the Gauss–Jordan method. As an eigenvalue solution method, the LR algorithm is applied; eigenvectors are computed via inverse iterations.

First, to establish the validity of our formulation, we will compare our results with available data. As a lossy anisotropic waveguide system, we consider a waveguide filled with a carbon-loaded rubber sheet useful as a wave absorber or a shielding material. The permittivity tensor is given by

$$\epsilon_{xx}=\epsilon_{xx}^{(0)}\cos^2\phi+\epsilon_{yy}^{(0)}\sin^2\phi \tag{40}$$

$$\epsilon_{yy}=\epsilon_{xx}^{(0)}\sin^2\phi+\epsilon_{yy}^{(0)}\cos^2\phi \tag{41}$$

$$\epsilon_{zz}=\epsilon_{zz}^{(0)} \tag{42}$$

$$\epsilon_{xy}=\epsilon_{yx}=\left(\epsilon_{xx}^{(0)}-\epsilon_{yy}^{(0)}\right)\sin\phi\cos\phi \tag{43}$$

$$\epsilon_{xz}=\epsilon_{zx}=\epsilon_{yz}=\epsilon_{zy}=0 \tag{44}$$

where ϕ is a rolling direction of the anisotropic sheet, $\epsilon_{xx}^{(0)}=11.86-j0.80$, $\epsilon_{yy}^{(0)}=20.83-j3.16$, and $\epsilon_{zz}^{(0)}=\epsilon_{xx}^{(0)}$ [8]. A schematic illustration of the guide is depicted in Fig. 2. When ϕ does not coincide with any coordinate axis, i.e., $\phi\neq n\cdot 90°$ for any integer n, the off-diagonal elements ϵ_{xy} and ϵ_{yx} are involved in the permittivity tensor. In this case, the earlier finite element formulation [5], [6] is of no use. Fig. 3(a) and (b) shows the dependence of the propagation constant on the rolling direction of the anisotropic sheet. It is found from these figures that our results are in excellent agreement with those obtained by the telegrapher equation method [8] both for phase and for attenuation.

Since the validity of the formulation has been demonstrated above, in what follows we will present some useful results computed by the present method.

Fig. 4(a)–(e) shows the dispersion curves for various rolling angles. As the permittivity tensor varies with ϕ, the profile of the curves is dependent on ϕ. It is interesting to note that in Fig. 4(c) and (d) the attenuation curve exhibits a dip at the transitional regime from the reactive to the dissipative region. Such behavior is inherent in an anisotropic system and has no counterpart in an isotropic system.

When one wants to calculate the reflection coefficients from lossy anisotropic material, which are necessary in measuring a complex permittivity tensor by a standing-wave method, detailed distributions of the electromagnetic field are needed. Fig. 5(a)–(c) and Fig. 6(a)–(c) display the magnetic field distributions for $\phi = 0$ and $\phi = 60°$, respectively. For $\phi = 0$, $H_y \equiv 0$ and the field is symmetric across both the x and the y axis, whereas for $\phi = 60°$, all three components of the magnetic field are involved and they exhibit no symmetry across the axes, indicating the full vectorial nature of the modal field.

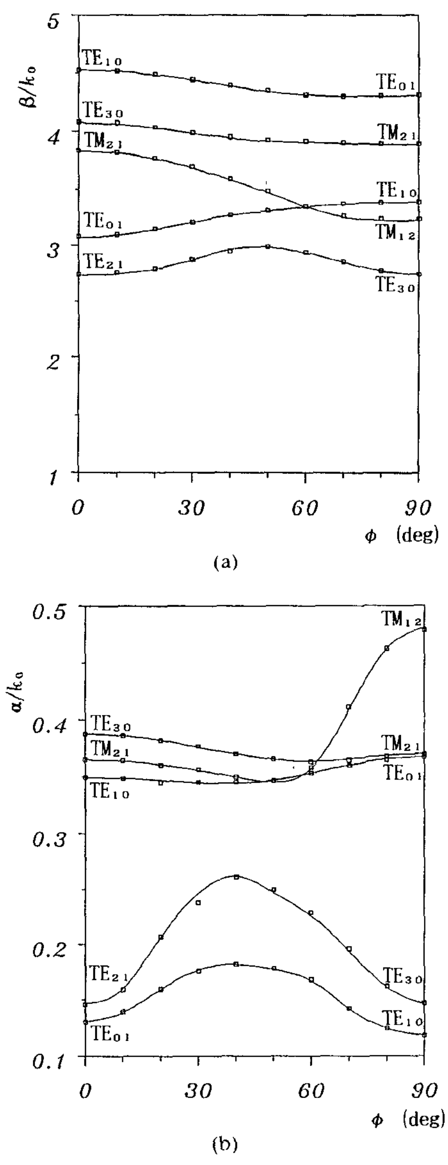

Fig. 3. Propagation constant versus rolling angle. $h/W = 0.4454$. $k_0 W = 4.5115$. $m+n =$ odd for TE_{mn} and TM_{mn} modes. The waveguide cross section is subdivided into 64 quadratic triangular elements; the number of nodal points is 153. Solid lines indicate the present results using (30); hollow squares indicate those obtained by Hashimoto *et al.* [8]. (a) Normalized phase constant. (b) Normalized attenuation constant.

IV. Concluding Remarks

We have developed a full vectorial finite element method for solving electromagnetic guided wave phenomena in lossy anisotropic media by extending the formalism in terms of the transverse magnetic field component established for lossy isotropic media. The main advantage of this approach is that one can avoid unnecessary iterations using complex frequencies because the eigenvalue of the final matrix equation corresponds to the propagation constant. To demonstrate the validity of the method, we have applied it to a rectangular waveguide filled with a carbon-loaded rubber sheet. As a result of its efficiency and versatility, we believe that the solution procedure described in this paper will be useful for designing wave absorbers, shielding materials, and other wave functional components utilizing lossy anisotropic materials.

Appendix I
The Explicit Form of Submatrices of $[C']$

The explicit form of submatrices of $[C']$ in the text is given by

$$[C^4_{xx}] = -[G_6^{yy}]$$
$$[C^4_{xy}] = [G_6^{yx}]$$
$$[C^4_{yx}] = [G_6^{xy}]$$
$$[C^4_{yy}] = -[G_6^{xx}]$$
$$[C^3_{xx}] = [G_5^{zy}] - [G_5^{yz}]^T$$
$$[C^3_{xy}] = -[G_5^{zx}] + [G_4^{yz}]^T$$
$$[C^3_{yx}] = -[G_4^{zy}] + [G_5^{xz}]^T$$
$$[C^3_{yy}] = [G_4^{zx}] - [G_4^{xz}]^T$$
$$[C^2_{xx}] = [G_2^{zz}] + \left(-[G_4^{yy}]^T + [G_5^{yx}]^T\right)[G_6]^{-1}[G_4]^T - [G_4][G_6]^{-1}\left([G_4^{yy}] - [G_5^{xy}]\right) + k_0 j(Z_n/Z_0)\sin^2\theta[G_6'] - k_0^2[G_6]$$
$$[C^2_{xy}] = -[G_3^{zz}]^T + \left(-[G_4^{yy}]^T + [G_5^{yx}]^T\right)[G_6]^{-1}[G_5]^T - [G_4][G_6]^{-1}\left(-[G_4^{yx}] + [G_5^{xx}]\right) - k_0 j(Z_n/Z_0)\sin\theta\cos\theta[G_6']$$
$$[C^2_{yx}] = -[G_3^{zz}] + \left([G_4^{xy}]^T - [G_5^{xx}]^T\right)[G_6]^{-1}[G_4]^T - [G_5][G_6]^{-1}\left([G_4^{yy}] - [G_5^{xy}]\right) - k_0 j(Z_n/Z_0)\sin\theta\cos\theta[G_6']$$
$$[C^2_{yy}] = [G_1^{zz}] + \left([G_4^{xy}]^T - [G_5^{xx}]^T\right)[G_6]^{-1}[G_5]^T - [G_5][G_6]^{-1}\left(-[G_4^{yx}] + [G_5^{xx}]\right) + k_0 j(Z_n/Z_0)\cos^2\theta[G_6'] - k_0^2[G_6]$$

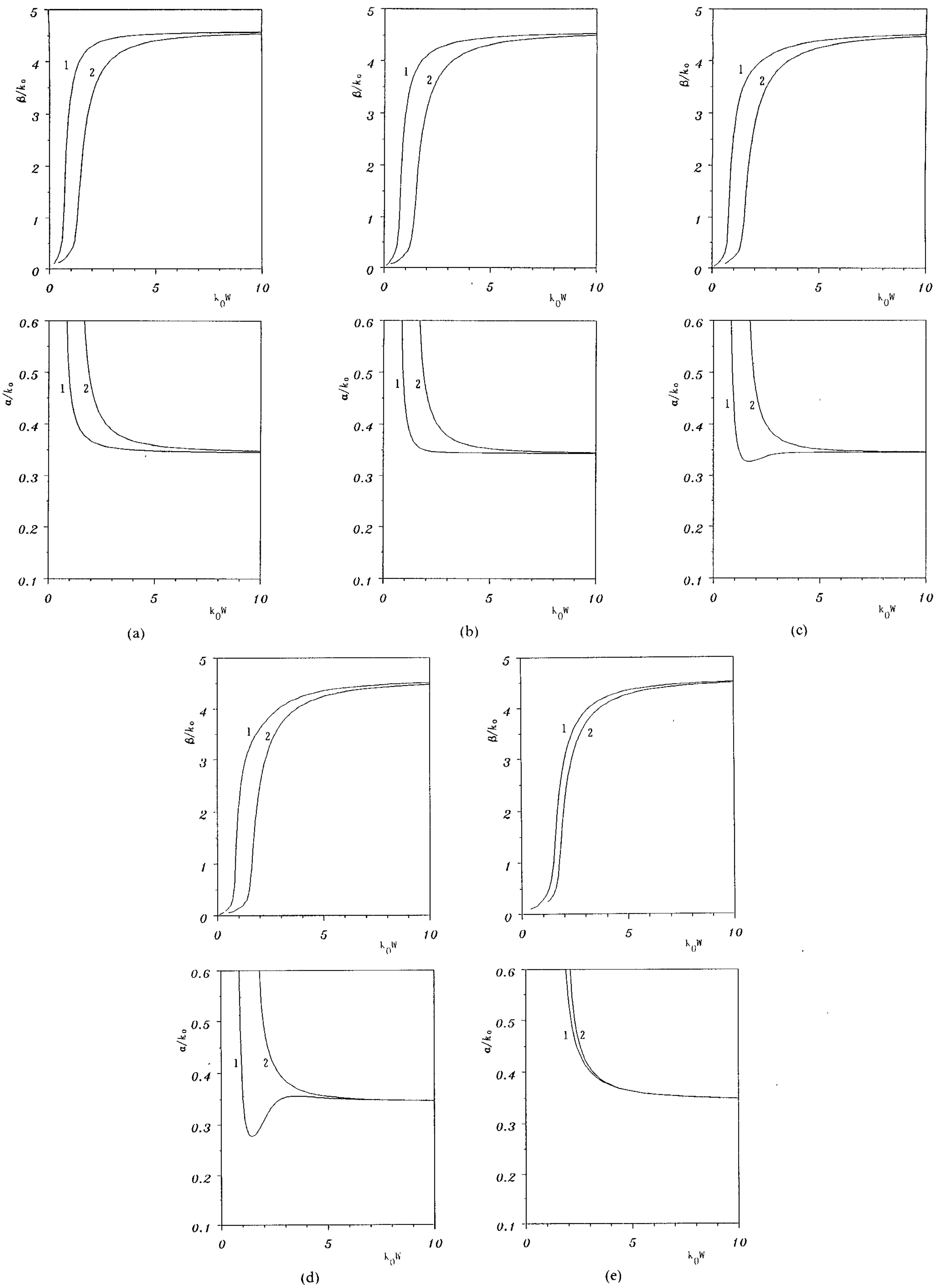

Fig. 4. Dispersion characteristics. $h/W = 0.4454$. Curves 1 and 2 correspond to the fundamental and the first higher order mode, respectively. Element division is the same as Fig. 3. (a) $\phi = 0$. (b) $\phi = 30°$ (c) $\phi = 45°$. (d) $\phi = 60°$. (e) $\phi = 90°$.

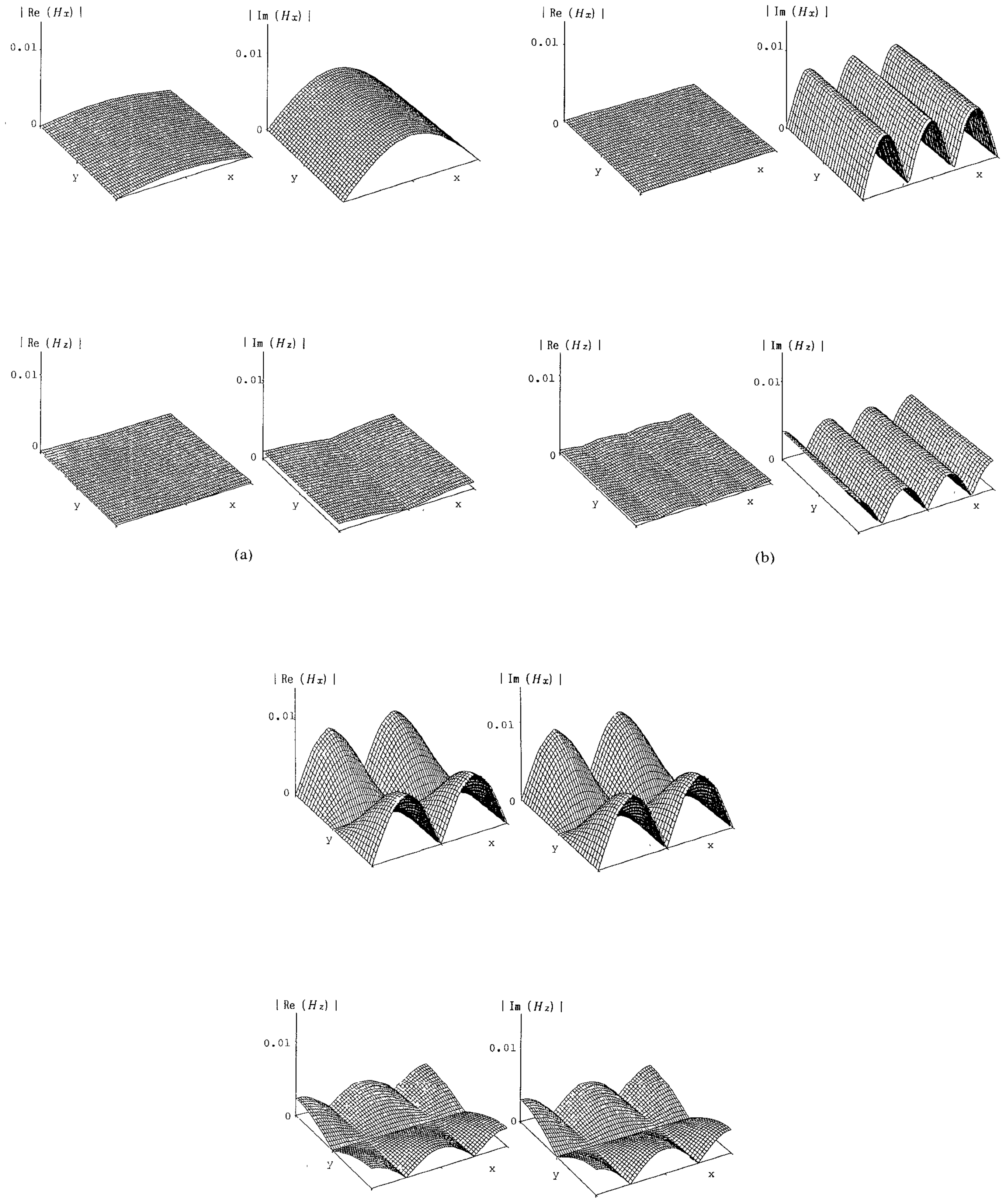

Fig. 5. Magnetic field distributions in cross section for $\phi = 0$. $h/W = 0.4454$. $k_0W = 4.5115$. $H_y = 0$ anywhere in cross section. (a) TE_{10} mode. (b) TE_{30} mode. (c) TM_{21} mode.

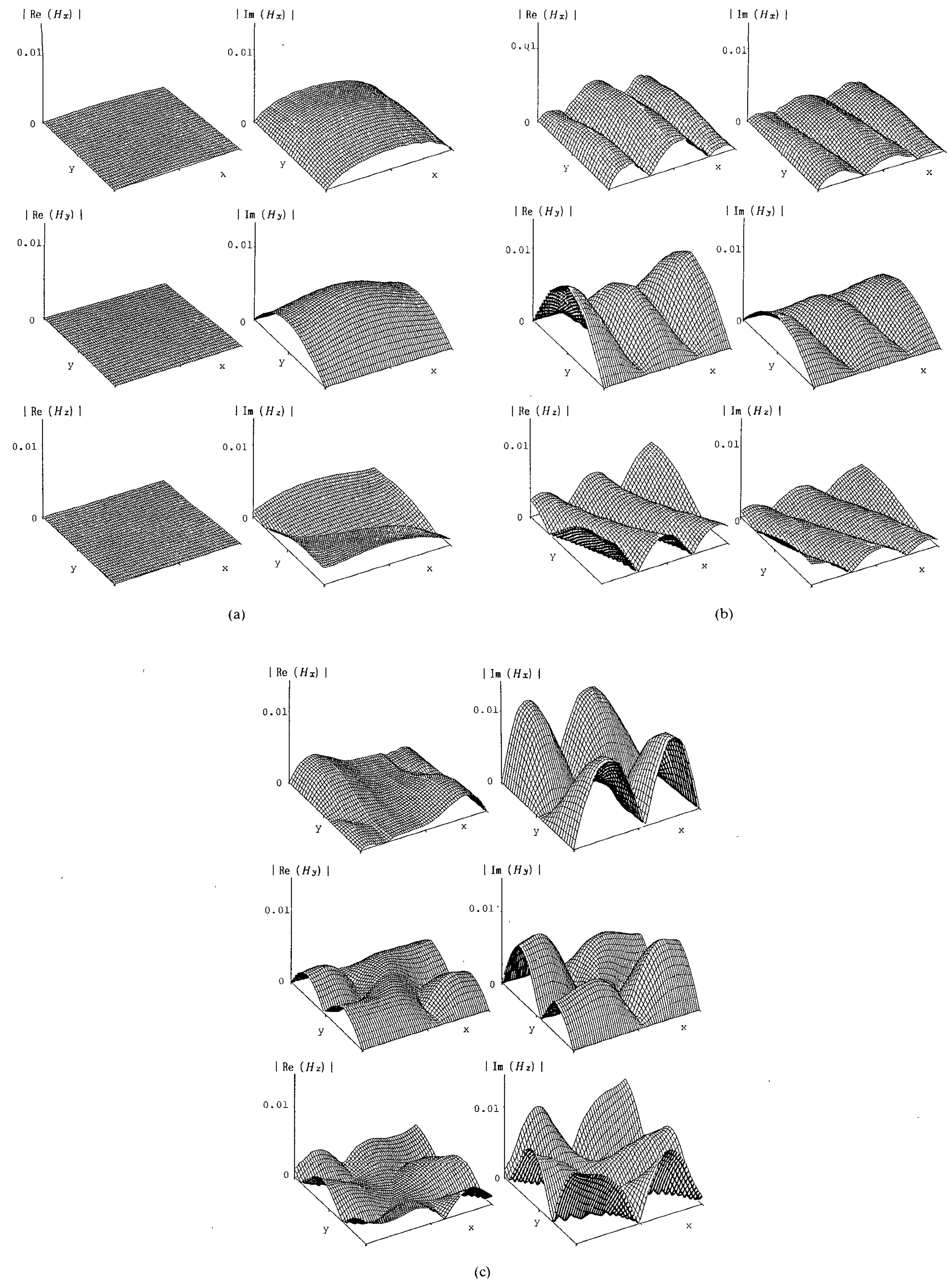

Fig. 6. Magnetic field distributions in cross section for $\phi = 60°$. $h/W = 0.4454$. $k_0W = 4.5115$. All three components are involved in this angle. The name of each mode corresponds to that for $\phi = 0$. (a) TE_{10} mode. (b) TE_{30} mode. (c) TM_{21} mode.

$$
\begin{aligned}
[C_{xx}^1] &= \left([G_3^{zy}]^T - [G_2^{zx}]\right)[G_6]^{-1}[G_4]^T \\
&\quad - [G_4][G_6]^{-1}\left([G_3^{yz}] - [G_2^{xz}]\right) \\
[C_{xy}^1] &= \left([G_3^{zy}]^T - [G_2^{zx}]\right)[G_6]^{-1}[G_5]^T \\
&\quad - [G_4][G_6]^{-1}\left(-[G_1^{yz}] + [G_3^{xz}]^T\right) \\
[C_{yx}^1] &= \left(-[G_1^{zy}] + [G_3^{zx}]\right)[G_6]^{-1}[G_4]^T \\
&\quad - [G_5][G_6]^{-1}\left([G_3^{yz}] - [G_2^{xz}]\right) \\
[C_{yy}^1] &= \left(-[G_1^{zy}] + [G_3^{zx}]\right)[G_6]^{-1}[G_5]^T \\
&\quad - [G_5][G_6]^{-1}\left(-[G_1^{yz}] + [G_3^{xz}]^T\right) \\
[C_{xx}^0] &= -[G_4][G_6]^{-1}\left([G_1^{yy}] - [G_3^{yx}]\right. \\
&\quad \left. - [G_3^{xy}]^T + [G_2^{xx}]\right)[G_6]^{-1}[G_4]^T \\
&\quad - k_0 j(Z_n/Z_0)[G_4][G_6]^{-1}[G_6'][G_6]^{-1}[G_4]^T \\
&\quad + k_0^2[G_4][G_6]^{-1}[G_4]^T \\
[C_{xy}^0] &= -[G_4][G_6]^{-1}\left([G_1^{yy}] - [G_3^{yx}]\right. \\
&\quad \left. - [G_3^{xy}]^T + [G_2^{xx}]\right)[G_6]^{-1}[G_5]^T \\
&\quad - k_0 j(Z_n/Z_0)[G_4][G_6]^{-1}[G_6'][G_6]^{-1}[G_5]^T \\
&\quad + k_0^2[G_4][G_6]^{-1}[G_5]^T \\
[C_{yx}^0] &= -[G_5][G_6]^{-1}\left([G_1^{yy}] - [G_3^{yx}]\right. \\
&\quad \left. - [G_3^{xy}]^T + [G_2^{xx}]\right)[G_6]^{-1}[G_4]^T \\
&\quad - k_0 j(Z_n/Z_0)[G_5][G_6]^{-1}[G_6'][G_6]^{-1}[G_4]^T \\
&\quad + k_0^2[G_5][G_6]^{-1}[G_4]^T \\
[C_{yy}^0] &= -[G_5][G_6]^{-1}\left([G_1^{yy}] - [G_3^{yx}]\right. \\
&\quad \left. - [G_3^{xy}]^T + [G_2^{xx}]\right)[G_6]^{-1}[G_5]^T \\
&\quad - k_0 j(Z_n/Z_0)[G_5][G_6]^{-1}[G_6'][G_6]^{-1}[G_5]^T \\
&\quad + k_0^2[G_5][G_6]^{-1}[G_5]^T.
\end{aligned}
$$

Here $[G_k]$, $[G_6']$, and $[G_k^{ij}]$ are defined by

$$
\begin{aligned}
[G_1] &= \sum_e \iint_e \{N\}_x \{N\}_x^T \,dx\,dy \\
[G_2] &= \sum_e \iint_e \{N\}_y \{N\}_y^T \,dx\,dy \\
[G_3] &= \sum_e \iint_e \{N\}_x \{N\}_y^T \,dx\,dy \\
[G_4] &= \sum_e \iint_e \{N\}_x \{N\}^T \,dx\,dy \\
[G_5] &= \sum_e \iint_e \{N\}_y \{N\}^T \,dx\,dy \\
[G_6] &= \sum_e \iint_e \{N\} \{N\}^T \,dx\,dy \\
[G_6'] &= \sum_{e'} \int_{e'} \{N\} \{N\}^T \,d\Gamma \\
[G_1^{ij}] &= \sum_e \iint_e \rho_{ij} \{N\}_x \{N\}_x^T \,dx\,dy \\
[G_2^{ij}] &= \sum_e \iint_e \rho_{ij} \{N\}_y \{N\}_y^T \,dx\,dy \\
[G_3^{ij}] &= \sum_e \iint_e \rho_{ij} \{N\}_x \{N\}_y^T \,dx\,dy \\
[G_4^{ij}] &= \sum_e \iint_e \rho_{ij} \{N\}_x \{N\}^T \,dx\,dy \\
[G_5^{ij}] &= \sum_e \iint_e \rho_{ij} \{N\}_y \{N\}^T \,dx\,dy \\
[G_6^{ij}] &= \sum_e \iint_e \rho_{ij} \{N\} \{N\}^T \,dx\,dy.
\end{aligned}
$$

Here ρ_{ij} is defined by

$$
\begin{bmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{yx} & \rho_{yy} & \rho_{yz} \\ \rho_{zx} & \rho_{zy} & \rho_{zz} \end{bmatrix} = [\epsilon]^{-1}
$$

$$
\begin{aligned}
\rho_{xx} &= (\epsilon_{yy}\epsilon_{zz} - \epsilon_{yz}\epsilon_{zy})/\Delta \\
\rho_{xy} &= (\epsilon_{xz}\epsilon_{zy} - \epsilon_{xy}\epsilon_{zz})/\Delta \\
\rho_{xz} &= (\epsilon_{xy}\epsilon_{yz} - \epsilon_{xz}\epsilon_{yy})/\Delta \\
\rho_{yx} &= (\epsilon_{yz}\epsilon_{zx} - \epsilon_{yx}\epsilon_{zz})/\Delta \\
\rho_{yy} &= (\epsilon_{xx}\epsilon_{zz} - \epsilon_{xz}\epsilon_{zx})/\Delta \\
\rho_{yz} &= (\epsilon_{xz}\epsilon_{yx} - \epsilon_{xx}\epsilon_{yz})/\Delta \\
\rho_{zx} &= (\epsilon_{yx}\epsilon_{zy} - \epsilon_{yy}\epsilon_{zx})/\Delta \\
\rho_{zy} &= (\epsilon_{xy}\epsilon_{zx} - \epsilon_{xx}\epsilon_{zy})/\Delta \\
\rho_{zz} &= (\epsilon_{xx}\epsilon_{yy} - \epsilon_{xy}\epsilon_{yx})/\Delta
\end{aligned}
$$

$$
\begin{aligned}
\Delta = {} & \epsilon_{xx}\epsilon_{yy}\epsilon_{zz} + \epsilon_{xy}\epsilon_{yz}\epsilon_{zx} + \epsilon_{xz}\epsilon_{yx}\epsilon_{zy} \\
& - \epsilon_{xz}\epsilon_{yy}\epsilon_{zx} - \epsilon_{xy}\epsilon_{yx}\epsilon_{zz} - \epsilon_{xx}\epsilon_{yz}\epsilon_{zy}.
\end{aligned}
$$

Appendix II
The Explicit Form of Submatrices of $[P^i]$

The explicit form of nonzero submatrices of $[P^i]$ in the text is given by

$$
\begin{aligned}
[P_{xx}^2] &= -[G_6^{yy}] \\
[P_{xy}^2] &= [G_6^{yx}] \\
[P_{yx}^2] &= [G_6^{xy}] \\
[P_{yy}^2] &= [G_6^{xx}] \\
[P_{zz}^2] &= -p^2[G_6] \\
[P_{xx}^1] &= -[G_5^{yz}]^T + [G_5^{zy}] \\
[P_{xy}^1] &= [G_4^{yz}]^T - [G_5^{zx}]
\end{aligned}
$$

$$[P_{xz}^1] = -j[G_4^{yy}]^T + j[G_5^{xy}]^T - jp^2[G_4]$$
$$[P_{yx}^1] = -[G_4^{zy}] + [G_5^{xz}]^T$$
$$[P_{yy}^1] = -[G_4^{xz}]^T + [G_4^{zx}]$$
$$[P_{yz}^1] = -j[G_5^{xx}]^T + j[G_4^{xy}]^T - jp^2[G_5]$$
$$[P_{zx}^1] = -j[G_4^{yy}] + j[G_5^{xy}] - jp^2[G_4]^T$$
$$[P_{zy}^1] = -j[G_5^{xx}] + j[G_4^{xy}] - jp^2[G_5]^T$$
$$[P_{xx}^0] = [G_2^{zz}] + p^2[G_1] - k_0^2[G_6] + j(Z_n/Z_0)k_0 \sin^2\theta [G_6']$$
$$[P_{xy}^0] = -[G_3^{zz}]^T + p^2[G_3] - j(Z_n/Z_0)k_0 \sin\theta\cos\theta [G_6']$$
$$[P_{xz}^0] = j[G_3^{zy}]^T - j[G_2^{xz}]^T$$
$$[P_{yx}^0] = -[G_3^{zz}] + p^2[G_3]^T - j(Z_n/Z_0)k_0 \sin\theta\cos\theta [G_6']$$
$$[P_{yy}^0] = [G_1^{zz}] + p^2[G_2] - k_0^2[G_6] + j(Z_n/Z_0)k_0 \cos^2\theta [G_6']$$
$$[P_{yz}^0] = -j[G_1^{zy}] + j[G_3^{zx}]$$
$$[P_{zx}^0] = -j[G_3^{yz}] + j[G_2^{xz}]$$
$$[P_{zy}^0] = j[G_1^{yz}] - j[G_3^{xz}]^T$$
$$[P_{zz}^0] = [G_1^{yy}] + [G_2^{xx}] - [G_3^{xy}]^T - [G_3^{yx}] - k_0^2[G_6]$$

where p is the penalty coefficient [10].

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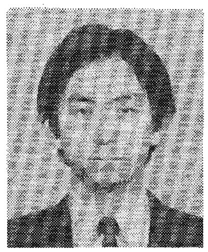

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